The Invariant Relations of Two Triangles.

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The Invariants of the Triangle

This paper will be divided into three sections, as follows:-

Sec. I. The determination of the invariants of the triangle in terms of its five fundamental invariants.

Sec. II. The case where the triangles [illegible] with integers.

Sec. III. The determination of the invariants of [illegible] such that it will permit the triangles in space, representing planes E, into triangles [illegible] [illegible].

Sec. I. The determination of the invariants of the triangles.

1. With the plane [illegible] of the

same order, one given in points and the other in lines, there is associated a triangle, such that the result of acting with the first polar of one of its lines as to the curve in lines, upon the curve [illegible] is [illegible] the line itself. And, similarly, for the points of the triangle.

Let the two curves be

$$a_x^n = 0, \quad [illegible] \quad u_\xi^n = 0.$$

The polar of a line, η, as to the line curve is

$$\eta_u\, u_\xi^{n-1} = 0.$$

Acting with this upon the point curve we have

$$\eta_a\, a_x^{n-1}\, x_\iota = 0.$$

If now we make the line identical to η, we have three equations of the type.

$$\frac{1}{n} a_\alpha^{n-1} \alpha_i \equiv \lambda y_i , \qquad (1).$$

These equations are homogeneous in the three quantities, y, and hence their determinant must vanish.

That is:

$$\Delta(\lambda) \equiv \begin{vmatrix} a_1 \alpha_1 a_\alpha^{n-1} - \lambda & a_2 \alpha_1 a_\alpha^{n-1} & a_3 \alpha_1 a_\alpha^{n-1} \\ a_1 \alpha_2 a_\alpha^{n-1} & a_2 \alpha_2 a_\alpha^{n-1} - \lambda & a_3 \alpha_2 a_\alpha^{n-1} \\ a_1 \alpha_3 a_\alpha^{n-1} & a_2 \alpha_3 a_\alpha^{n-1} & a_3 \alpha_3 a_\alpha^{n-1} - \lambda \end{vmatrix} = 0.$$

$$\equiv -\left(\lambda^3 + I_1 \lambda^2 + I_2 \lambda + I_3\right). \qquad (2).$$

The roots of this equation, substituted in the equations (1), will give the three lines, y.

From the symmetric way in which a and α enter into 2, we see that, had we required the points, y, such that the polars of any one of them as to the point curve acting upon the

their curve, give the point itself, the condition for consistency of the equations corresponding to (1) would have been the same as (2), the rows and columns of the determinant being merely interchanged.

On the assumption that the equation (2) has three distinct roots, it can be shown that the points, y, are the intersections of the three [illegible].*

We have the two identities,

$$y_a\, a_x^{n-1}\, a_b = \lambda\, y_b$$

$$y'_\alpha\, \alpha_a^{n-1}\, a_\xi = \lambda'\, y'_\xi .$$

where λ and λ' are supposed distinct roots.

Operate with y' on the first of these

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* Cf. "Zur [illegible] [illegible]", [illegible] David Hilbert; Math. Ann., vol. 28, p. 403.

identities and with y_b on the second.
This gives
$$y_a \, a_\alpha^{n-1} \, y'_\alpha \equiv \lambda \, y'_y$$
$$y'_\alpha \, \alpha_a^{n-1} \, y_a \equiv \lambda' \, y'_y.$$

The left sides are equal, and hence
either $\lambda = \lambda'$ or $y'_y = 0$. But, by our hypothesis, the former is not the case, and so y' lies on the line y. Similarly, the point y'', corresponding to the root λ'', lies on the line y. Hence the three points y belong to the same line; e.

2. We shall now consider the case when $n = 3$, and the point and line cubics degenerate respectively into three lines and three points.
Let the 3-point be

And the 3-line

$$\lambda_i \ldots y_i = 0.$$

The polar of a line, y, as to the 3-point is

$$\sum_a a_\gamma b_\beta [illegible] = 0,$$

and the polar of this conic as to the 3-line is

$$\sum_a \sum_\alpha a_\gamma (b_\beta c_\gamma + b_\gamma c_\beta) \alpha_i = 0.$$

Upon making this line identical to y, we have these equations of the type

$$\sum_a \sum_\alpha (b_\beta c_\gamma + b_\gamma c_\beta) a_\gamma \alpha_i = \lambda y_i \qquad (3).$$

We shall now adopt the following notation.

$$b_\beta c_\gamma + b_\gamma c_\beta \equiv (bc|\beta\gamma).$$

$$\sum_a (bc|\beta\gamma)\, a_i \equiv (\beta\gamma)_i$$
$$\sum_\alpha (bc|\beta\gamma)\, \alpha_i \equiv (bc)_i$$
$$\sum_\alpha (\beta\gamma)_i\, \alpha_j \equiv \sum_a (bc)_j\, a_i \equiv B_{ji}$$

The equation (3) then becomes

$$\sum_a \sum_\alpha (bc|\beta\gamma)\, a_y\, \alpha_i - \lambda y_i = 0.$$

Or,

$$\sum_a (bc)_i\, a_y - \lambda y_i = 0.$$

And, finally,

$$\sum_j B_{ij}\, y_j - \lambda y_i = 0. \qquad (3').$$

The three equations of this type are consistent if

$$\Delta(\lambda) \equiv \begin{vmatrix} B_{11}-\lambda & B_{12} & B_{13} \\ B_{21} & B_{22}-\lambda & B_{23} \\ B_{31} & B_{32} & B_{33}-\lambda \end{vmatrix} = 0.$$

This equation may be written in the form

$$\lambda^3 + I_1 \lambda^2 + I_2 \lambda + I_3 = 0, \qquad (4).$$

where

$$I_1 = -\sum_1^3 B_{11}, \quad I_2 \equiv \sum_1^3 B_{22} B_{33} - B_{23} B_{32}$$

$$I_3 = -\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} (bc)_1 & (bc)_2 & (bc)_3 \\ (ca)_1 & (ca)_2 & (ca)_3 \\ (ab)_1 & (ab)_2 & (ab)_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \cdot \begin{vmatrix} (bc|\beta\gamma) & (bc|\gamma\alpha) & (bc|\alpha\beta) \\ (ca|\beta\gamma) & (ca|\gamma\alpha) & (ca|\alpha\beta) \\ (ab|\beta\gamma) & (ab|\gamma\alpha) & (ab|\alpha\beta) \end{vmatrix},$$

= [illegible]

Where

$D = |abc|$, $\Delta = |\alpha\beta\gamma|$

and N is the third determinant above with the negative sign.

3. The coëfficients of the Equation (4) are the invariants of a collineation arising from the 3-line and the 3-point. The double points of this collineation are given by the equation (3'), and the collineation itself by

Y: $\quad y_i' = \sum_j B_{ij} y_j$.

The corresponding line collineation, formed by acting with the polar of η_k as to the 3-point upon the 3-line, is

H: $\quad \eta_i' = \sum_j B_{ji} \eta_j$.

On the hypothesis that the roots of (4)

are distinct, we have shown that the fixed lines of H are the joins of the fixed points of Y. The triangle, formed by these fixed points and lines, may then be taken as the triangle of reference. In this case

$$\beta_{ij} = 0, \quad j \neq i.$$

The collineations then become

Y: $\quad x_i' = \beta_{ii} x_i,$

H: $\quad y_i' = \beta_{ii} y_i.$

The point collineation, equivalent to H, evidently is

$$x_i' = \beta_{ii}' x_i.$$

It thus appears that __H is the inverse of Y.__

Let now the triangle of reference

taken as the triangle formed by the e-point instead of that of the fixed points

Then,

$$B_{i1} = \sum_{\alpha} (\beta_2\gamma_3 + \beta_3\gamma_2)\,\alpha_i \,, \text{ etc..}$$

The collineation

$$Y: \qquad y_i' = B_{i1}\,y_1 + B_{i2}\,y_2 + B_{i3}\,y_3$$

Sends the point $(1\,0\,0)$ into (B_{11}, B_{21}, B_{31}). This point lies on the line $(1\,0\,0)$ if B_{11} is zero.

But,

$$\tfrac{1}{3} I_1 = B_{11} = B_{22} = B_{33} = \sum_{\alpha} (\beta_2\gamma_3 + \beta_3\gamma_2)\,\alpha_1 \,.$$

If then the invariant I_1 vanishes, the collineation Y sends the e points into the invariant triangle; Y_{12}, thus formed from the symmetrical way in which

the 3-line and the 3-point enter into the equations of the collineations Y and H, we see that the vanishing of I_1 is also the condition that H sends the 3-line into a circumscribed triangle.

It, then, follows from the fact that H is the inverse of Y that if I_1 vanishes the collineation Y sends the 3-point and the 3-line into inscribed triangles, and that the collineation H sends them into circumscribed triangles.

If the invariant I_2 vanishes, the sums of the minors of the elements of the principal diagonals in the determinants of Y and H are zero. Such combinations may be said to be sub-normal, and they have the property of sending certain triangles into circumscribed and inscribed triangles, respectively. It

in common, [illegible] [illegible] [illegible] [illegible]. The 3-point and the 3-line is one of these.

If I_3 vanishes, the determinants of Y and H are zero. These collineations have, then, no inverses.

In this case, one of the roots of the equation

$$\Delta(\lambda)=0 \qquad (4).$$

is zero.

This zero root, instead of giving a proper fixed point of the collineation Y, will give a point, y_0, which the collineation explodes; i.e. which Y sends anywhere. Similarly this root gives a line η_0 which H explodes.

In general y_0 and η_0 are not incident point and line. They may, then, be taken as the point $(1\,0\,0)$ and the line [illegible] respectively.

In this case, we have

$$B_{i1} = B_{1i} = 0, \qquad \text{for } i = 1, 2, 3.$$

The collineations then become

$$Y: \begin{cases} y_1' = 0, \\ y_2' = B_{22} y_2 + B_{23} y_3, \\ y_3' = B_{32} y_2 + B_{33} y_3, \end{cases}$$

and

$$H: \begin{cases} \eta_1' = 0, \\ \eta_2' = B_{22} y_2 + B_{32} y_3, \\ \eta_3' = B_{23} y_2 + B_{33} y_3. \end{cases}$$

It is, then, evident that Y sends all points on a line through y_0 into the same point on γ_0, and H sends all lines through a point on γ_0 into the same line through y_0. There will be no lines through y_0 such that all points on each of them will be sent by Y into.

[illegible] point and the axis y_0. [illegible] point [illegible] [illegible] [illegible] [illegible] to the two proper fixed points of Y. Take these as the points $(0,1,0)$ and $(0,0,1)$.

The collineations then become

$$Y: \begin{cases} y_1' = y_1 \\ y_2' = \rho_{22}\, y_2 \\ y_3' = \rho_{33}\, y_3 \end{cases}$$

and

$$H: \begin{cases} y_1' = 0 \\ y_2' = B_{22}\, y_2 \\ y_3' = B_{33}\, y_3 . \end{cases}$$

Then, as we should expect, the fixed lines of H are the joins of the fixed points of Y to the point y_0.

It is easily shown that, if Y sends y into y', then H sends any line through y' into the join of y and y_0.

since

$$I_3 \equiv -\Delta D N,$$

the collineations may be brought into three forms when the 3-point or the 3-line degenerates.

It may be shown that, if the 3-line degenerates, Y_0 is the point in which the three lines meet, and if the 3-point degenerates, Y_0 is the line on which the three points lie.

The meets of the 3-line and the joins of the 3-point form a new 3-point and 3-line; from which we can build up two collineations Y' and H'. The question arises whether the fixed points of the collineations Y and Y' are the same.

We shall not attempt to answer this question here, but shall only remark that the problem reduces to whether or not it is possible to find a point such that the pencil of conics, formed by the [illegible] to two arbitrary 3-lines, shall contain two conics, one apolar to the one 3-line and the other to the other.

4. [illegible] of the equation (4), with reference to the 3-point and 3-line themselves.

It will be found useful to know the forms which the invariants assume when the 3-line is taken as the triangle of reference, and the point [illegible]

For this end we shall put

$$\alpha_1 = \beta_2 = \gamma_3 = 1,$$

$$\alpha_2 = \alpha_3 = \beta_3 = \beta_1 = \gamma_1 = \gamma_2 = 0,$$

$$a_k = r_k e^{i\theta_k}, \quad c_k = r_k e^{-i\theta_k},$$

Where r_k is the length of the side k of the triangle of reference, and θ_k the angle which an arbitrary line makes with that side.

If we denote by α_i the interior angles of the triangle, by c_i the cotangents of these angles, and take the area of the triangle as unity, we have the following identities,

$$\alpha_1 = \pi - (\theta_2 - \theta_3),$$

$$e^{i(\theta_2 - \theta_3)} = -\cos\alpha_1 + i\sin\alpha_1,$$

$$r_2 r_3 \sin\alpha_1 = 2,$$

$$r_2 r_3 \cos\alpha_1 = 2c_1 .$$

$c_2c_3 + c_3c_1 + c_1c_2 = 1.$

Hence, follows

$b_2c_3 + b_3c_2 = -4c_1,$

$b_2c_3 - b_3c_2 = 4i,$

$b_1c_1 = 2(c_2 + c_3).$

The invariants then take the following form.

$$I_1 = -\sum B_{11} = -\sum_\alpha \sum_\beta (bc|\beta\gamma)\, a_\alpha$$

$$= -3 \sum_1 (b_2c_3 + b_3c_2)\, a_1$$

$$= 12 \sum c_1 a_1 . \qquad (5).$$

[illegible]

Evidently,

$$B_{11} = B_{22} = B_{33} = -4 \sum c_1 a_1 ,$$

$$B_{23} = 4\left[-2c_2a_3 + (c_1+c_2)a_1\right],$$

$$B_{32} = 4\left[-2c_3a_2 + (c_3+c_1)a_1\right].$$

[illegible]:

$$\sum B_{22} B_{33} = 48 \sum c_1^2 a_1^2 + 2 c_2 c_3 a_2 a_3$$

and

$$\sum B_{23} B_{32} = 16 \sum (1+c_1^2) a_1^2 + 4 (2 c_2 c_3 - 1) a_2 a_3 .$$

We have, then,

$$I_2 = 16 \sum (2 c_1^2 - 1) a_1^2 + 2 (2 - c_2 c_3) a_2 a_3 . \qquad (6).$$

$$\Delta = |\alpha \beta \gamma| = i . \qquad (7).$$

$$D = \sum (b_2 c_3 - b_3 c_2) a_1 = 4 i \sum a_1 \qquad (8).$$

$$1' = - \sum (b_2 c_3 + b_3 c_2) \Big[(c_3 a_1 + c_1 a_3)(a_1 b_2 + a_2 b_1) - (c_1 a_2 + c_2 a_1)(a_3 b_1 + a_1 b_3) \Big] ,$$

$$= - \sum (b_2^2 c_3^2 - b_3^2 c_2^2) a_1^2 - \Big[(b_3 c_1 + b_1 c_3)(b_1 c_2 - b_2 c_1) + (b_1 c_2 + b_2 c_1)(b_3 c_1 - b_1 c_3) \Big] a_2 a_3$$

$$= 16 i \sum c_1 a_1^2 - (c_2 + c_3) a_2 a_3 \qquad (9).$$

Finally, the equation (9) expresses the condition that o is on the ninepoint circle of the triangle; i.e. on the circle through the middle points of the sides and the feet of the perpendiculars.

From (5), (7) and (8), we have

$$\Delta_1 I_1 = 48 i \sum c_1 a_1^2 + (c_2+c_3) a_2 a_3. \quad (10).$$

5. If the vanishing of some function of the invariants Δ, D, I_1, I_2 and N be found to express the condition that the point and line bear a certain projectively invariant property, then the vanishing of the same function of a set of invariants, formed by substituting in the original invariants, for the Italics the minors of the corresponding Greek letters in a

and for the Greek letters those of the corresponding Italics in D, must express the condition that the meets of the 3-line and the joins of the 3-point have the same invariant properties.

We shall give the forms taken by the invariants Δ, D, I_1 and N upon making this substitution, but shall not perform the actual work of determining them. If the accented letters denote the invariants in which the substitution has been made, we shall have

$$\left.\begin{aligned} I_1' &= -\tfrac{1}{2}(\Delta D I_1 - 9N), \\ \Delta' D' &= \Delta^2 D^2, \\ N' &= \frac{\Delta^2 D^2}{6}(\Delta D I_1 + 3N). \end{aligned}\right\} \quad (11).$$

As an example of the use that may be made of these equations [illegible]

work out the following problem.

Required the locus of points, a, such that the Feuerbach circle of the triangle a·b·c shall pass through a point, k, where b, c and k are given fixed points.

If b and c are the Absolute, we have seen that

$$N = 0$$

expresses the condition that a lies on the Feuerbach circle of the 3-line. If, then, α be the line at infinity and β and γ the circular rays through k, the equation

$$N' = 0$$

expresses the condition that k lies on the Feuerbach circle of a·b·c. Hence, if a [illegible] point, this equation [illegible]

required locus of a. If neither the 3-point nor the 3-line degenerates, a lies on a conic which we shall see later is the conic through b and c and apolar to $\alpha\beta\gamma$, (see 5, III). Assuming this, K is evidently the centre of the conic and the asymptotes are perpendicular.

6. We are now in a position to determine the covariants of the 3-line and 3-point.

These covariants will be denoted by Roman numerals. The covariant, formed from a given one by the substitution (1), will be denoted by the same numeral with an accent, and that, formed by an interchange of the words point

in the interpretation of the invariant, by the numeral with an asterisk.

I. The 3-point degenerates if

$$P = 0. \qquad \text{I.}$$

I.* The 3-line degenerates if

$$\Delta = 0. \qquad \text{I.}^*$$

II. Let the 3-point be apolar to the 3-line.

Then, $a_\xi b_\xi c_\xi$ acting on $\alpha_x \beta_x \gamma_x$ must be zero.

That is,

$$\sum_a \sum_\alpha (bc)(\beta\gamma) \alpha_a = \sum B_{11} = 0.$$

But, this is, by definition, I_1.

The apolarity condition is then

[illegible]

By an interchange of the Greek and Italic letters in II, we have the condition that the 3-line is apolar to the 3-point; i.e. the invariant II*. But this interchange does not alter II, and so, as we know from the geometrical character of the conditions, the invariants II and II* are the same.

II′. <u>That the joins of the 3-point be apolar to the meets of the 3-line.</u>

The required condition may be found, to within powers of Δ and D, from II by the equations (11). We shall, however, find it independently.

If the joins of the 3-point are apolar to the meets of the 3-line, the joins $\overline{ca}$ and $\overline{ab}$ are apolar to the polar lines of [illegible]

is, if b and c are the absolute and the 3-line the triangle of reference, the circular rays through a are apolar to the polar conic of the line at infinity as to the triangle. This conic is, we know, the maximum inscribed ellipse

$$\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2 = 0.$$

The lines aI and aJ are apolar to this conic, and hence the tangents from a are perpendicular. Then a lies on the director circle of the maximum inscribed ellipse.

If we define the intermediate of two line conics as the locus of points, a, such that the lines of the two conics through them form harmonic pairs, the director circle is evidently the

intermediate of the Ellipse and the Absolute.

The intermediate of two conics,

$$\pi_\xi^2 = 0 \quad \text{and} \quad \omega_\xi^2 = 0,$$

is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}^2 = 0.$$

or, expanded,

$$\Sigma(\pi_{22}\,\omega_{33} + \pi_{33}\,\omega_{22} - \pi_{23}\,\omega_{23})\,a_1^2 + (\pi_{31}\,\omega_{12} + \pi_{12}\,\omega_{31} - \pi_{11}\,\omega_{23} - \pi_{23}\,\omega_{11})\,a_2\,a_3 = 0.$$

Let

$$\pi_\xi^2 \equiv \Sigma\, \xi_2\,\xi_3 \qquad \text{and} \quad \omega_\xi^2 \equiv \Sigma\,(e_2+e_3)\,\xi_1^2 - 2\,e_1\,\xi_2\,\xi_3.$$

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1. Salmon, Conic Sections [illegible]

Substituting these values in the equation of the intermediate, we have

$$\sum c_1 a_1^2 - 2(c_2+c_3)\, a_2 a_3 = 0,$$

as the equation of the director circle.

From the equations [illegible] and [illegible] we have

$$\Delta D I_1 - 9N = -96 i \sum c_1 a_1^2 - 2(c_2+c_3)\, a_2 a_3.$$

Then, the condition required is

$$\Delta D I_1 - 9N = 0. \qquad \text{II}'.$$

If two points of the 3-point or two lines of the 3-line coincide, II$'$ is identically satisfied. But, in this case, the join of the coincident points or the meet of the coincident lines is arbitrary, and may be so taken that the points of the 3-point shall be apolar to the meets of the 3-line.

II. Let there be a conic circumscribed to the 3-point and apolar to the 3-line. When the 3-line is the triangle of reference and b and c the Absolute, the conic becomes the apolar circle.

We must, then, have

$$\sum c_1 a_1^2 = 0.$$

From the equations (9) and (10), it follows that

$$\Delta D I_1 + 3N = 96 i \sum c_1 a_1^2.$$

The condition then becomes

$$\Delta D I_1 + 3N = 0. \qquad \text{III.}$$

It remains to see if the degree of this invariant is correct.

A point conic apolar to the 3-line

is of the class -

$$m_1(\alpha_b)^2 + m_2(\beta_b)^2 + m_3(\gamma_b)^2 = 0.$$

This passes through the points a, b and c if

$$\left|(\alpha_a)^2, (\beta_b)^2, (\gamma_c)^2\right| = 0.$$

This determinant is of the same degree in the Italic and Greek letters as is III, which is, then, the invariant required.

Evidently III is also the condition that there be a conic inscribed in the 3-line and apolar to the 3-point. For, an interchange of the Italic and Greek letters leaves it unaltered.

III′. If there be a conic circumscribing the [illegible] of the 3-point and apolar to the 3-line.

The conic becomes the apolar parabola whose focus is at a, when the 3-line is the triangle of reference and b and c the Absolute.

To aid in finding the required invariant, we shall show that the joins of the middle points of the sides of the triangle are lines of the parabola.

Any apolar parabola is of the form

$$\Sigma m_1 \xi_1^2 = 0, \qquad \text{where } \Sigma m_1 = 0.$$

The middle point of the side ([illegible]) is

$$\xi_1 + \xi_3 = 0.$$

Eliminate ξ_1 between the equation of this point and that of the parabola. The resulting equation gives the points in which the tangents from the

middle point of the side (100) cut the side (001).

These points are given, then, by the equation

$$m_1\xi_1^2+(m_2+m_3)\xi_2^2=0.$$

(b). Since $\Sigma m_1 = 0$, eq.

$$\xi_1^2-\xi_2^2=(\xi_1+\xi_2)(\xi_1-\xi_2)=0.$$

Then the tangents from the middle point of one side of the triangle cut the other two sides in their middle points and in their infinite points. Hence, the joins of the middle points are lines of the apolar parabola.

But, a circle, circumscribing a triangle formed by three tangents to a parabola, passes through the focus.[1]

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1. Salmon, Conic Sections. p. 207.

Then the circle through the middle points of the sides passes through the point. That is, the point lies on the Feuerbach circle of the triangle. The equation of this circle, we have seen, is

$$W = 0.$$

This, then, is the required condition, provided the 5-line and 5-point do not degenerate.

Let a', b' and f' denote the minors in Δ of the corresponding small letters.

A line conic apolar to the 5-line is then of the form

$$\sum_1^5 m_1 (\alpha_5')^2 = 0.$$

The 6-point conic apolar to the 5-point is of the form

[illegible] $= 0.$

If we make these two conics identical, we have six equations, homogeneous in the six undetermined coëfficients. The determinant of the equations must then vanish. This determinant is, evidently, the invariant we are trying to find. It is, however, of the sixth degree in the Italic and accented Greek letters, and consequently, of the twelfth in the original Greek letters. The invariant N is of the sixth degree in both sets of letters. It must then include Δ^2, as this is the only invariant of the sixth degree in the Greek letters.

The required condition, then, is

$$\Delta^2 N = 0. \qquad \text{III}.$$

This invariant may be verified by applying the equations (11) to III.

Since the invariant contains the factor Δ, it vanishes identically for a degenerate 3-line. That is, it can only put two conditions on a conic to make it apolar to a degenerate 3-line. That this is so is immediate, since the conic $a_x^2 = 0$ is apolar to the 3-line $b_2 b_3 (b_2 + b_3) = 0$, if

$$a_{22} = a_{33} = - a_{23} .$$

There are, then, but two conditions put upon the conic and so it can be made to touch any three lines.

It would, perhaps, appear to be always possible to determine a conic apolar to a given 3-line and touching the joins of three points on a line. That is, it might seem that the factor

" should enter into the required invariant. That this is not the case we can easily show.

For, let the 3-point be

$$\xi_2 \xi_3 (\xi_2 + \xi_3) = 0.$$

A conic, touching the joins of these points, is of the form

$$m_1 \xi_2 \xi_3 + (\xi_2 + \xi_3)(m_2 \xi_2 + m_3 \xi_3) = 0.$$

There are but two independent coefficients and so the conic can be made to satisfy but two other conditions.

II. Is there a conic circumscribed to the 3-point and to the 3-line.

When the 3-line is the fundamental triangle and b and c the Absolute, the conic becomes the circum-circle.

That is, we must have

$$\Sigma (c_2+c_3) a_2 a_3 = 0.$$

From the equations (9) and (10), we have

$$\Delta D I_1 - 3N = 96 i \,\Sigma (c_2+c_3) a_2 a_3.$$

The condition, then, becomes

$$\Delta D I_1 - 3N = 0.$$

To investigate the degree of the required condition, we proceed as follows. The condition that the six points α', β', γ', a, b, c lie on a conic is the vanishing of the [illegible] of the six equations formed by substituting, in the general equation of a conic, the coördinates of three points. This determinant is of the sixth degree in the accented Greek letters and in the italic, and hence

of the twelfth degree in the original Greek letters.

We must then introduce the factor Δ^2.

Hence, the required invariant becomes

$$\mathfrak{D}^2(\Delta \mathfrak{D} I_1 - 3N') = 0. \qquad \text{IV.}$$

IV* [illegible] inflexional to the 3-line and to the 3-point.

The required invariant is formed by an interchange of the Italic and Greek letters in IV.

It is, then,

$$D^2(\Delta D I_1 - 3N) = 0. \qquad \text{IV}^*$$

If neither the 3-line nor the 3-point degenerates, the conditions IV and IV* are the same.

We have, then, an immediate proof of the well known theorem; that if

vertices of two triangles, circumscribed to a conic, lie on a conic.[1]

V. Let there be a line conic apolar to both the 3-line and 3-point.

A line conic apolar to the 3-line is of the form

$$\Sigma\, m_1 (\alpha'_\xi)^2 = 0.$$

And one apolar to the 3-point is of the form

$$\Sigma\, m_4 (a_\xi)^2 = 0.$$

If these conics are identical, we have six equations of the type

$$\Sigma\, m_1 \alpha'_i \alpha'_j = \Sigma\, m_4\, a_i a_j.$$

---

1. Salmon, Conic Sections, pp. 320, 343.

The determinant of these equations must vanish. But this determinant is the same as that which was found in investigating the degree of the invariant II, the rows and columns being merely interchanged.

The required invariant is, then

$$\Delta^2(\Delta D I_1 - 3N) = 0. \qquad \text{V.}$$

Similarly, there is a point conic apolar to the 3-line and to the 3-point if

$$D^2(\Delta D I_1 - 3N) = 0. \qquad \text{V.}^*$$

VI. <u>Let there be a point, y, such that its polar conic as to the 3-line is apolar to the 3-point.</u>

The polar conic of y as to the 3-line is

[illegible]

This conic is apolar to the 3-point, if
$$\sum_a \sum_\alpha (bc|\beta\gamma)\,\alpha_y\, a_\xi \equiv 0.$$

We must, then, have three equations of the type
$$\sum_a \sum_\alpha (bc|\beta\gamma)\,\alpha_y\, a_i = 0.$$

or, what is evidently the same thing,
$$B_{1i}\,y_1 + B_{2i}\,y_2 + B_{3i}\,y_3 = 0.$$

The condition that these equations are consistent is
$$I_3 \equiv \Delta D I_1 = 0. \qquad \underline{VI.}$$

This is of course, also the condition that there is a line whose polar conic as to the 3-point is apolar to the 3-line.

$\underline{VII}$ [illegible]

is polar conic as to the points of the point is opposite to the (points of the) 3-line.

The required invariant may be found by applying the equations (II) to the invariant VI.

It is, then,

$$\Delta D\,(\Delta D I_1 + 3N) = 0 . \qquad I I'$$

VII. (a) The three polar lines as to the 3-line, of the points of the 3-point, taking two at a time, meet in a point.

The polar line of $b_1 c_1$ as to the 3-line is

$$\sum (bc/\beta\gamma)\, x_\alpha = \sum_1^3 (bc)_1\, x_1 = 0 .$$

These three lines meet in a point if

$$|(bc)_1, (ca)_2\ (ab)_3| \equiv \Delta N = 0 . \qquad VII$$

This gives, then, a simple projective definition of the Feuerbach conic of two points and a triangle, where we mean, by the Feuerbach conic, that conic which will project into the Feuerbach circle of the triangle when the two points are projected into the Absolute.

VIII. Let there be a collineation having fixed points at a, b, c, which sends each of the vertices of the 3-line into a point on the opposite line. We shall first show that, if b and c are sent into the Absolute, the locus of a is the orthocentroidal circle of the 3-line.[1]

---

1. This fact was pointed out by Dr. Morley in his lectures on Geometry during the year [illegible].

In the proof of this preliminary theorem, we shall make use of the system of circular coördinates. That is, a point of the plane will be expressed by two conjugate imaginaries, p and $\bar{p}$, such that

$$p = X + iY,$$
$$\bar{p} = X - iY$$

where X and Y are the rectangular Cartesian coördinates of the point.

The circular coördinates of all points on the axis of X will be real, while those of points on that of Y will be pure imaginaries. The axes will then be called the axis of reals and the axis of imaginaries, respectively. It is evident that p carries with it its conjugate. The value of p is, then, sufficient to determine the point

a point on the unit circle, we have

$$|z| = z\bar{z} = X^2 + Y^2 = 1. \qquad (12).$$

In order to let z always define a definite point, the convention is made that there is but a single point at infinity.

The bilinear substitution

$$z' = \frac{az+b}{cz+d},$$

sends a circle into a circle,[1] provided a straight line be considered a circle through the infinite point.

If then infinity be a fixed point of the substitution, circles and straight lines are sent into circles and straight lines. A substitution of this type is termed [illegible]

1. Harkness and Morley, Intro. Anal. Fcts., p. 30.

upon the points of the projective plane which leaves the absolute unaltered.

The necessary and sufficient condition that [illegible] be a fixed point is that

$$c = 0.$$

Let the vertices of the 3-line be the points $a_1\, a_2\, a_3$, and so take the axes that these points shall lie on the unit circle and that the axis of reals shall be the Euler line of the triangle formed by them.

The orthocentre is then

$$S = a_1 + a_2 + a_3,$$

[illegible]

[illegible]

Since the inner line is, by definition, the join of these points, S is real.

We wish now to show that, if a substitution, having fixed points at infinity and g, sends $a_1 a_2 a_3$ into an inscribed triangle, then the point g lies on the ortho-centroidal circle of the triangle of the points $a_1 a_2 a_3$.

A collineation, having fixed points at infinity and g, is of the form

$$z' = k(z-g)+g. \qquad (13).$$

If a_1 is to be sent into a point, a_1', on the join of a_2 and a_3, we must have the equation

$$a_1' = \frac{a_2+\lambda a_3}{1+\lambda},$$

[illegible]

$$a_1'(1+\lambda) = a_2 + \lambda a_3 . \qquad (14).$$

If $\bar{k}$ and $\bar{g}$ denote the conjugates of k and g, the conjugate of (13) is

$$\bar{t}' = \bar{k}(\bar{t} - \bar{g}) + \bar{g} .$$

Hence, we have the value of $\bar{a}_1'$,

$$\bar{a}_1' = \bar{k}(\bar{a} - \bar{g}) + \bar{g} . \qquad (15).$$

Since a_1 is a point on the unit circle,

$$\bar{a}_1 = 1/a_1 .$$

Also, since λ is real, it equals its conjugate, and so the conjugate of (14) is, upon clearing of fractions,

$$a_2 a_3 \bar{a}_1'(1+\lambda) = a_3 + \lambda a_2 . \qquad (16).$$

Adding (14) and (16) and dividing by $(1+\lambda)$, we have

$$\bar{a}_1' a_2 a_3 + a_1' = a_2 + a_3.$$

Upon replacing $\bar{a}_1'$ and a_1' by their values found from the equations (13) and (15), this equation becomes

$$a_2 a_3 [\bar{k}(\bar{a}_1 - \bar{f}) + \bar{f}] + k(a_1 - f) + f - a_2 - a_3 = 0.$$

or,

$$a_2 a_3 [\bar{k}(1 - a_1 \bar{f}) + a_1 \bar{f}] + a_1 k (a_1 - f) + a_1 f - a_1(a_2 + a_3) = 0.$$

And, finally,

$$a_2 a_3 \bar{k} - s_3 \bar{k} \bar{f} + s_3 \bar{f} + k a_1 (a_1 - f) + a_1 f - a_1 (a_2 + a_3) = 0, \quad (17).$$

where $s_3 = a_1 a_2 a_3$.

By a cyclical interchange of the a's, we have the condition that a_2' lies on the join of a_3 and a_1.

This gives us

$$a_3 a_1 \bar{k} - s_3 \bar{k} \bar{f} + s_3 \bar{f} + k a_2 (a_2 - f) + a_2 f - a_2 (a_3 + a_1) = 0, \quad (18).$$

Subtract (18) from (17) and divide by the factor $(a_1 - a_2)$. We, then, have

$$-a_3\bar{K} + K(a_1+a_2) + f(1-K) - a_3 = 0. \qquad (19).$$

If we cyclically interchange the a's in this equation, we have

$$-a_1\bar{K} + K(a_2+a_3) + f(1-K) - a_1 = 0. \qquad (20).$$

From the equations (19) and (20), we can eliminate any one of the quantities K, $\bar{K}$ or f.

Eliminating in turn f and $\bar{K}$, we have the two relations

$$K + \bar{K} + 1 = 0, \qquad (21).$$

and

$$f = \frac{-sK}{1-K}. \qquad (22).$$

The equation (21) puts a condition on the collineation which is independent of the triangle considered. A collinea- tion which [illegible] this condition may

be said to be normal.[1] From the equation (21), we see that the real part of κ is minus one half.

The equation (22) is a bilinear substitution, and, hence, if κ runs along a line or circle, ζ runs along a line or circle. But, we have just seen that κ runs along the line $-1/2$. For the imaginary part of κ zero and infinity, ζ takes the values 5 and 5/3. The point ζ runs along a

---

1. It may be easily shown that the condition that a collineation on the points of the projective plane is normal is that the sum of the elements of the leading diagonal of the determinant of the collineation is zero. See page 11.

circle and not a line since f cannot be infinite as long as the real part of k is $-1/2$. Then f moves along a circle through the centroid and orthocentre.

The bilinear substitution sends inverse points into inverse points. The centre of the circle on which f moves is the inverse, in the f-plane, of the point at infinity. For f infinite, k is unity. Then the point in the k-plane, corresponding to the centre, is the inverse of unity in the line $-1/2$. That is, it is the point -2. Then, for k equal to -2, f is the centre. The centre is, then, $2s/3$ or the middle point of the join of the centroid and orthocentre.
[illegible]

centroidal circle of the triangle.

It remains then to express, in terms of our fundamental invariants, the equation of this circle.

The ortho-centroidal circle is one of the pencil of circles determined by the apolar- and circum- circles. That is, it is one of the pencil given by the equation

$$\Delta D I_1 + \lambda N = 0.$$

From (9) and (10), we have

$$\Delta D I_1 + \lambda N \equiv 10 i \sum (3+\lambda) c_1 a_1^2 + (3-\lambda)(c_2+c_3) a_2 a_3.$$

This circle passes through the [illegible] (1,1,1), if

$$\sum (3+\lambda) c_1 + (3-\lambda)(c_2+c_3) = 0.$$

$$(9-\lambda)(c_1+c_2+c_3)=0.$$

Then $\lambda = 9$, and the required invariant takes the form

$$\Delta D I_1 + 9N = 0. \qquad \text{VIII}.$$

This is evidently also the condition that there be a collineation (of lines), leaving fixed each of the lines of the 3-line, which sends the joins of the 3-point each into a line through the opposite point.

We shall now show that VIII is also the condition that there be a collineation leaving fixed points at the meets of the 3-line, which sends each point of the 3-point into a point on the join of the other two.

The equations (11), applied to VIII, will evidently give us the required invariant.

From then we have

$$\Delta' D' I_1' + 9\Lambda' = \Delta^2 D^2 (\Delta D I_1 + 9\Lambda).$$

The condition is then, upon excluding the cases of degenerate 3-point and 3-line, the same as VIII.

We have, then, the following theorem: —

If there exist a collineation, having fixed points at a, b and c, which sends each of the points A, B and C into a point on the join of the other two, then there also exists a collineation, with fixed points at A, B and C, which sends each of the points a, b and c into a point on the join of

The Sixth Case.

Here the vanishing of the invariant VIII expresses the fact that there is a mutual relation between the two triangles, considered as two 3-lines or as two 3-points.

The only invariants of the type

$$\Delta \mathcal{J} I_1 + \lambda N = 0$$

which are reproduced to within a factor by the substitutions of the equations (11), are easily seen to be those for which λ equals -3 and 9. The former expresses the condition that the points of the two triangles lie on a conic. This is evidently a mutual relation among the six points. The invariant arising from the value 9 is,

On the other hand, a mutual relation of the two triangles and not of the six points.

The results of Sec. (1) may be tabulated as follows.

The 3-point degenerates, if $\Delta = 0$.	The 3-line degenerates, if $\Delta = 0$.
The 3-point is apolar to the 3-line, if $I_1 = 0$.	The 3-line is apolar to the 3-point, if $I_1 = 0$.
The meets of the 3-line are apolar to the joins of the 3-point, if $\Delta D I_1 - 9N = 0$.	The joins of the 3-point are apolar to the meets of the 3-line, if $\Delta D I_1 - 9N = 0$.

A conic, circumscribed to the 3-point and apolar to the 3-line, exists, if

$$\Delta D I_1 + 3N = 0.$$

A conic, inscribed to the 3-line and apolar to the 3-point, exists, if

$$\Delta D I_1 + 3N = 0.$$

A point conic, apolar to the 3-line and to the 3-point, exists, if

$$D^2(\Delta D I_1 - 3N) = 0.$$

A line conic, apolar to the 3-point and to the 3-line, exists, if

$$\Delta^2(\Delta D I_1 - 3N) = 0.$$

A conic, circumscribed to both the 3-point and 3-line, exists, if

$$\Delta D I_1 - 3N = 0$$

A conic, inscribed to both the 3-line and 3-point exists, if

$$\Delta D I_1 - 3N = 0.$$

Hence exists a point whose polar conic as to the 3-line is apolar

Hence exists a line whose polar conic as to the 3-point is apolar

is the 3-point, if

$$\Delta D N = 0.$$

The three polar lines, as to the 3-line, of the points of the 3-point taking two at a time, meet in a point, if

$$\Delta N = 0.$$

There exists a collineation, having the 3-point as fixed points, which sends each point of the 3-line into a point on the opposite line, if

$$\Delta D I_1 + 9N = 0.$$

is the 3-line, if

$$\Delta D N = 0.$$

The three polar points, as to the 3-point, of the lines of the 3-line, taking two at a time, join on a line, if

$$\Delta N = 0.$$

There exists a collineation, having the 3-line as fixed lines, which sends each join of the 3-point into a line through the opposite point if $\Delta D I_1 + 9N = 0$

There exists a collineation, having the meets of the 3-line as fixed points, which sends each point of the 3-point into a point on the join of the other two, if

$$\Delta D I, + 9N = 0.$$

There exists a correlation, having the joins of the 3-point as fixed lines, which sends each line of the 3-line into a line through the meet of the other two, if

$$\Delta D I, + 9N = 0.$$

Sec. (2). The Case when the triangles [illegible] both ways.

1. If a be considered the variable point, the equations

$$\Delta D I, - 3N = 0$$

and

$$\Delta D I, + 3N = 0$$

represent respectively a circumconic and an apolar conic of the 3-line, $\alpha\beta\gamma$.

These conics meet in four points I, J, K and L which pair off with two points on the line

$$\delta = 0,$$

and two on the line

$$\varepsilon = 0.$$

The former pair may, of course, be the points at infinity of the fundamental triangle. For the moment, they will be considered as the points I and J.

If, then, a coincides with either K or L, it forms with the points I and J a triangle which is apolar both ways to that formed by the 3-line. For, evidently, in this case both I.

and I_1' vanish.

The triangles KIJ and LIJ are, then, apolar to the 3-line.

An interchange of the pairs I, J and K, L evidently interchanges the lines

$$D = 0 \quad \text{and} \quad I_1 = 0.$$

Then,

$$[illegible] = 0$$

represents the locus of all points, a, such that the 3-point aKL is apolar to $\alpha\beta\gamma$. Hence, IKL and JKL are both apolar to the [illegible] line.

It then [illegible] that the points I, J, K and L are such that a 3-point, formed from any three of them, is apolar to the 3-line, $\alpha\beta\gamma$.

As, in the case [illegible], the polar of any two of the points I, J, K and L [illegible]

[illegible] is the join of the other two: i.e. these four points form a <u>conjugate 4-point</u> of the 3-line.

2. We shall now show that the [illegible] conic of the 3-line, K and L is the same as that of the 3-line, I and J. For this it will be convenient to know the actual coordinates of K and L, when I and J are the Absolute and the 3-line the triangle of reference.

The points K and L are the intersections of the line

$$I_1 = 0$$

and the polar conic

$$\Delta D I_1 + 3N = 0.$$

These equations, in the case considered, become

and

$$\Sigma c_i a_i^2 = 0$$

respectively.

We now consider these equations as homogeneous in the quantities $c_i a_i$. Then, upon solution, we have

$$c_1 a_1 : c_2 a_2 : c_3 a_3 = \begin{Vmatrix} a_1 & a_2 & a_3 \\ 1 & 1 & 1 \end{Vmatrix}.$$

And, hence

$$\left.\begin{aligned} \rho c_1 a_1 &= a_2 - a_3 \\ \rho c_2 a_2 &= a_3 - a_1 \\ \rho c_3 a_3 &= a_1 - a_2 \end{aligned}\right\} \quad (23).$$

These equations are consistent, if

$$\begin{vmatrix} \rho c_1 & 1 & -1 \\ -1 & \rho c_2 & 1 \\ 1 & -1 & \rho c_3 \end{vmatrix} = 0$$

That is,

$$\rho^2 c_1 c_2 c_3 + \rho(c_1 + c_2 + c_3) = 0.$$

The root zero gives, from the equations (23), a point which does not lie on the line or on the conic.

The two others give the two points of intersection

$$\rho = \pm\sqrt{-\frac{c_1+c_2+c_3}{c_1c_2c_3}}.$$

Solving the equations (23), two at a time, we have

$$\begin{aligned} a_1 : a_2 : a_3 &= \rho^2 c_2 c_3 + 1 : 1 - \rho c_3 : 1 + \rho c_2 \\ &= 1 + \rho c_3 : \rho^2 c_3 c_1 + 1 : 1 - \rho c_1 \\ &= 1 - \rho c_2 : 1 + \rho c_1 : \rho^2 c_1 c_2 + 1. \end{aligned}$$

Adding, the coördinates are given in the symmetric form

$$a_1 = 3 + c_2 c_3 \rho^2 - (c_2 - c_3)\rho.$$

If we let ρ have the positive sign, the coördinates of K and L are

$$K_1 = 9 + c_2 c_3 \rho^2 + (c_2 - c_3)\rho$$

and

$$L_1 = 9 + c_2 c_3 \rho^2 - (c_2 - c_3)\rho.$$

The Feuerbach conic of the 3-line, K and L is then found by substituting these values of K and L for b and c in the equation

$$-N \equiv \sum (b_2 c_3 + b_3 c_2)(b_2 c_3 - b_3 c_2)\, a_1^2 - [(b_3 c_1 + b_1 c_3)(b_1 c_2 - b_2 c_1) + (b_1 c_2 + b_2 c_1)(b_3 c_1 - b_1 c_3)]\, a_2 a_3.$$

From the expressions for K_i and L_i we have

$$K_2 L_3 = 3c_1(c_2 + c_3)\rho^2 + 3(c_2 + c_3 - 2c_1)\rho + 9 + c_1^2 c_2 c_3 \rho^4$$
$$-(c_3 c_1 + c_1 c_2 - c_2 c_3 - c_1^2)\rho^2 + c_1(2c_2 c_3 - c_3 c_1 - c_1 c_2)\rho^3.$$

Evidently the value of $K_3 L_2$ is found by

changing the sign of ρ.

Then

$$K_2L_3 + K_3L_2 = 18 + 2c_1^2c_2c_3\rho^4 - 2\{c_3c_1 + c_1c_2 - c_2c_3$$
$$- c_1^2 - 2c_1c_2 - 2c_3c_1\}\rho^2.$$

But,

$$\rho^4 = -\frac{c_1+c_2+c_3}{c_1c_2c_3}\rho^2$$

and

$$c_2c_3 + c_3c_1 + c_1c_2 = 1.$$

Hence,

$$K_2L_3 + K_3L_2 = 18 + 2\{-c_1(c_1+c_2+c_3) + c_1^2 - c_2c_3 + 2\}\rho^2$$
$$= 18 + 2\rho^2. = 2(9+\rho^2).$$

Similarly,

$$K_2L_3 - K_3L_2 = -2\rho(9+\rho^2)c_1$$

Then, the Feuerbach conic of the 3-line K and L becomes

$$\Sigma c_1\alpha_1^2 - (c_2+c_3)\alpha_2\alpha_3 = 0$$

But this is the Feuerbach circle

lines.

Hence, it follows that <u>the Feuerbach Conic of the 3-line, K and L is the same as that of the 3-line, I and J.</u>

Since the Feuerbach circle passes through the middle points of the sides and through the feet of the perpendiculars, it follows that <u>the foot of the perpendicular from a vertex upon the opposite side and the point where that side cuts the line KL correspond to the other two sides, and that a harmonic with the side it bisects are harmonic to K and L.</u>

3. Since the conic

$$\Delta D I_1 + 9 N = 0$$

passes through the points ... I and

L, there exists a collineation, having fixed points at any three of these points, which sends the 3-line into an inscribed triangle.

We shall now show that the triangle into which the collineation sends the 3-line, is in perspective with the 3-line, and that the center of perspective is the point into which the collineation sends the remaining one of the points I, J, K, L.[1]

---

1. This fact was suggested by a theorem of Georg Majcen in the Archiv d. Math. u. Physik, Oct. 16, 1902. On pages 94-95, he shows that, if we project the points of a triangle, ABC, from an intersection of the circum- and apollon-circles, into a new triangle $A_1^*B_1^*C_1^*$, the triangle will [illegible]

As in Sec. (1), VIII, we shall make use of the system of circular coördinates.

Let the vertices of the 3-line be the points $a_1\, a_2\, a_3$ on the unit circle, and let the Euler line of this triangle be taken as the axis of reals.

We wish now to find the coördinates of the intersections of the [illegible] and apolar circles.

The Feuerbach circle passes through these points, and so we can determine them as the intersections of this circle and the unit circle.

If

$$s = a_1 + a_2 + a_3,$$

the [illegible]

[illegible]

sends each of the points a_1, a_2, a_3 into the middle point of the join of the other two. It, then, sends the unit circle into the Feuerbach circle. That is, the map-equation of the Feuerbach circle is

$$y = -\tfrac{1}{2}(t - s)$$

where t runs along the unit circle.

If, also,

$$|y| = 1$$

the point y is on the unit circle.

Then the points required are the points t satisfying the equation

$$|y| = \left|\tfrac{1}{2}(t - s)\right| = 1,$$

whence

[illegible]

The conjugate of y is

$$\bar{y} = -\frac{1}{2}\left(\frac{1}{t} - s\right).$$

Then,

$$y\bar{y} = \frac{1}{4}(t-s)\left(\frac{1}{t} - s\right) = 1.$$

Or,

$$(t-s)(1-st) = 4t,$$

$$st^2 + (3-s^2)t + s = 0.$$

Then,

$$t = \frac{s^2-3 \pm i\sqrt{10s^2 - s^4 - 9}}{2s}.$$

Let $B = 1-s^2$ and $A \equiv \sqrt{10s^2 - s^4 - 9} \equiv \sqrt{-B(B+8)}$.

We have, then,

$$t = \frac{-2-B \pm iA}{2s}.$$

---

1. A is real or imaginary according as the circum- and apolar circles do or do not intersect in real points, i.e. according [illegible]

[illegible]

The points K and L are then found by substituting these values of t in the equation

$$y^2 = 4(x-s).$$

They are, then

$$K = \frac{1}{4s}(4 - B + iA),$$

and $$L = \frac{1}{4s}(4 - B - iA).$$

We wish now to have a collineation, having fixed points at K and ∞, which sends $a_1\, a_2\, a_3$ into an inscribed triangle. Any collineation, L, having fixed points at K and ∞, is of the form

$$L: \qquad p' = \lambda(p - K) + K.$$

From the equations (1) and (2) of page 51, we have

$$\lambda + \bar{\lambda} + 1 = 0$$

and $\kappa = -$ [illegible]

Solving the latter equation for λ, we have:

$$\lambda = \frac{\kappa}{\kappa - s}.$$

The collineation is then,

$$L: \qquad x' = \frac{\kappa}{\kappa - s}(x - \kappa) + \kappa.$$

or,

$$L: \qquad x' = \frac{\kappa}{\kappa - s}(x - s).$$

Now,

$$\frac{\kappa}{\kappa - s} = \frac{4 - B + iA}{4 - B + iA - 4s^2}.$$

This reduces to

$$\lambda = \frac{\kappa}{\kappa - s} = \frac{-B + iA}{2B}.$$

Evidently, the condition

$$\lambda + \bar{\lambda} + 1 = 0$$

is satisfied.

The collineation, then, becomes

or:
$$z' = \frac{-B+iA}{2B}\,(z-s). \qquad (28)$$

The point L is sent into

$$L' = \frac{-B+iA}{2B}\left(\frac{4-B-iA}{4s} - s\right)$$

$$= \frac{-B-2+iA}{2s}.$$

The point a_1 is sent into

$$a_1' = \frac{-B+iA}{2B}\,(a_1 - s).$$

The points a_1, a_1' and L' lie on a line

if
$$\frac{a_1' + \lambda a_1}{1+\lambda} = L'$$

for λ real, . .

Substitution for a_1' and L' [illegible]

$$\frac{-\frac{B+iA}{2B}(a_1-s)+\lambda a_1}{1+\lambda} = \frac{-B-2+iA}{2s}.$$

Or,

$$s\left[(-B+iA)(a_1-s)+2B\lambda a_1\right] = B(1+\lambda)(-B-2+iA).$$

Collecting for a_1, we have

$$a_1 s\left[B(2\lambda-1)+iA\right] = s^2(-B+iA)+B(1+\lambda)(-B-2+iA)$$

$$= -B\lambda(B+2-iA)-3B+iA.$$

The conjugate equation is

$$\frac{s}{a_1}\left[B(2\bar\lambda-1)-iA\right] = -B\bar\lambda(B+2+iA)-3B-iA.$$

Multiplying these two equations, we have

$$s^2\left[B^2\{4\lambda\bar\lambda-2(\lambda+\bar\lambda)+1\}+A^2+2iAB(\bar\lambda-\lambda)\right] =$$

$$B^2\lambda\bar\lambda(4+4B+B^2+A^2)+9B^2+A^2$$

$$+3B^2\left[(2+B)(\lambda+\bar\lambda)+iA(\bar\lambda-\lambda)\right]$$

$$-iAB\left[(2+B)(\bar\lambda-\lambda)+iA(\lambda+\bar\lambda)\right].$$

The coefficient of $\lambda\bar{\lambda}$ is

$$B^2(4S^2-4-4B-B^2-A^2)=0.$$

That of $(\lambda+\bar{\lambda})$ is

$$-2S^2B^2-3B^2(2+B)-A^2B=$$
$$B^2[2(B-1)-3(2+B)+B+8]=0.$$

That of $(\bar{\lambda}-\lambda)$ is

$$iAB(2S^2-3B+2+B)=4iAB(1-B).$$

Finally, the term free of λ and $\bar{\lambda}$ is

$$S^2(B^2+A^2)-9B^2-A^2=(1-B)(B^2-B^2-8B)$$
$$-9B^2+B^2+8B$$
$$=0.$$

The equation, then, becomes

$$4iAB(1-B)(\bar{\lambda}-\lambda)=0$$

Hence λ is equal to the conjugate and therefore is real

that is, the line joining a_1 and a_1' passes through L'.

The triangles $a_1 a_2 a_3$ and $a_1' a_2' a_3'$ are then in perspective and the center of perspective is the point L'. Also all similar ones as is a fixed point of L.

The inverse of the collineation L is the collineation

$$L': \qquad p' = \frac{13+iA}{4}\, p + S, \qquad (25)$$

Sends $a_1 a_2 a_3$ into a circumscribed, similar and perspective triangle, $a_1'' a_2'' a_3''$. This collineation sends $a_1 a_2 a_3$ and $a_1' a_2' a_3'$ into $a_1'' a_2'' a_3''$ and $a_1 a_2 a_3$, and L' into L. Since L is a fixed point lines are sent into lines. Then the center of perspective of the triangles $a_1 a_2 a_3$ and $a_1'' a_2'' a_3''$ is the point L.

Further, L^{-1} sends the circumcenter, O,

with the collineation, so that one fixed point S is the centre of the circle with which C' [illegible] the circum-circle. And the fixed point A is one of the intersections of the circum- and polar circles, and S is the centre of the polar circle. Then, the circle with which u' [illegible] the circum-circle of $u_1 u_2 u_3$ is the polar circle of the triangle. That is, the circum-circle of $u_1'' u_2'' u_3''$ is the polar circle of $u_1 u_2 u_3$.

In [illegible], then, [illegible] Theorem, [illegible] [illegible] 10 where k is the point [illegible] and $u_1 u_2 u_3$ and $u_1'' u_2'' u_3''$ are respectively ABC and $A_1^* B_1^* C_1^*$.

<u>Thus, exists then a collineation having fixed points at three of the intersections</u> of [illegible] circum- and polar.

[illegible] of a given triangle, which sends the triangle into a circumscribed perspective triangle, the centre of perspective being at the fourth point, and the conic through the fixed points and circumscribed to the same triangle being apolar to the old triangle. The inverse of this collineation sends the triangle into an inscribed perspective triangle, the centre of perspective being the point into which the fourth intersection of the two conics is sent.

Similarly, there is a collineation having fixed points at the [illegible] of [illegible] tice, which sends the triangle formed by any three of the points I, J, K and L, say that IJK, into a circumscribed perspective triangle, the cen-

<u>of perspective having the fourth point of intersection of the conic circumscribed to the 3-line, I, J, K and L, and of the conic circumscribed to the 3-line and apolar to I J K.</u>

4. It may happen that the collineation, having fixed points at three of the points I, J, K, L, say at I, J and K, which sends the 3-line into an inscribed triangle, is such that its cube is the identical collineation.

That is,

$$L^3 = 1.$$

Or, what is the same thing,

$$L^2 = L^{-1}.$$

From the equations (24) and (25), we

have L and L^{-1} in the forms

$$L: \qquad b' = \frac{-B+iA}{2B}(b-S), \qquad (24).$$

and

$$L^{-1}: \qquad b' = \frac{B+iA}{4}b + S. \qquad (25).$$

Then, further,

$$L^2: \qquad b' = -\frac{B+iA}{2B}\left[\frac{-B+iA}{2B}(b-S) - S\right]$$
$$= \frac{1}{2B}\left[(B+4-iA)b - 4S\right].$$

If, then,

[illegible]

we have

$$\frac{1}{2B}\left[(B+4-iA)b - 4S\right] \equiv \frac{B+iA}{4}b + S,$$

for all values of b.

That is,

$$\left.\begin{aligned} &2(B+4-iA) = B(B+iA), \\ \text{and}& \\ &-2S = SB. \end{aligned}\right\} \qquad (26).$$

The second of these two equations

given

$$B = -2 \quad \text{or} \quad A = 0.$$

The first of the equations (26), upon equating the real and imaginary parts, gives the two relations

$$2B + 8 = B^2, \quad \text{or} \quad (B+2)(B-4) = 0,$$

and

$$-2A = BA, \quad \text{or} \quad A(B+2) = 0.$$

Hence the necessary and sufficient conditions in order that the cube of L shall be the identical collineation are

$$(B+2)(B-4) = 0,$$

$$A(B+2) = 0,$$

and either $B = -2$ or $A = 0$.

[illegible]

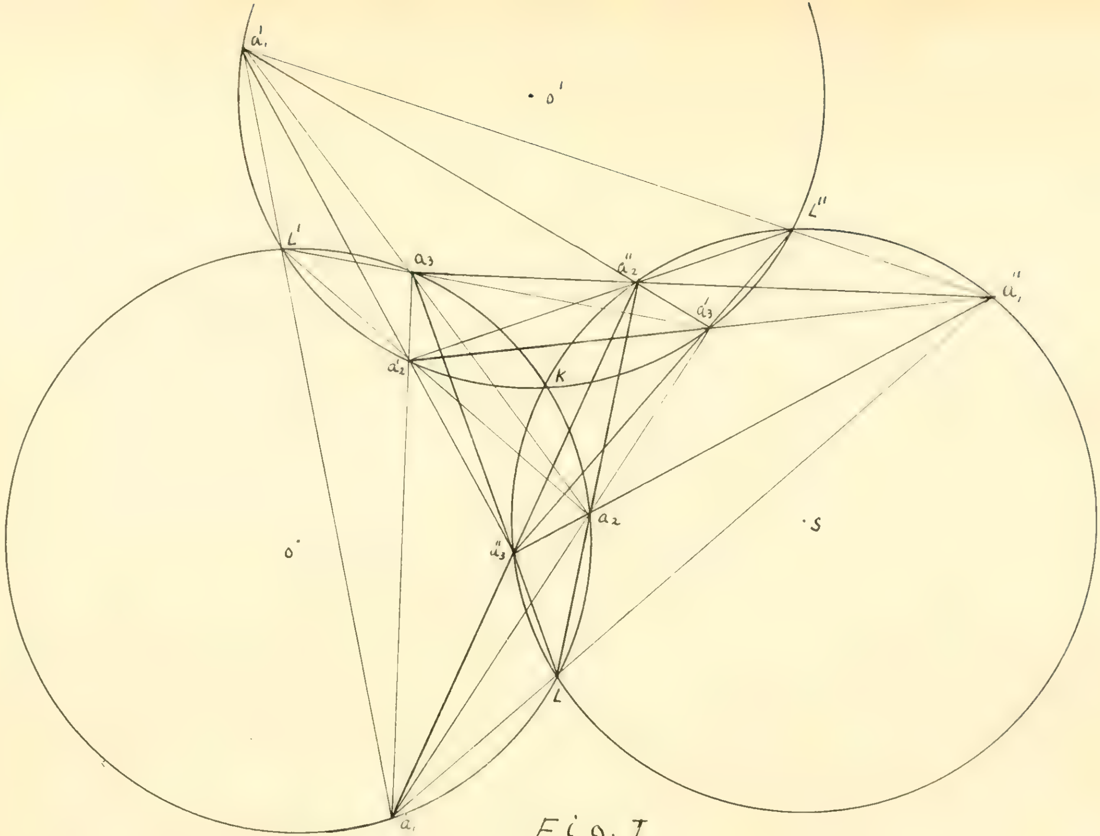

Fig. I.

Since s may be taken as positive, $s = \sqrt{3}$.

Thus, evidently, when $s = \sqrt{3}$, the collineations, L, L^2 and L^3 send $a_1 a_2 a_3$ respectively into $a_1' a_2' a_3'$, $a_1'' a_2'' a_3''$ and $a_1 a_2 a_3$. These three triangles are, then, such that each is circumscribed to[I] and in perspective with the one following, and since infinity is a fixed point, they are all similar. See Fig. I

Upon making $s = \sqrt{3}$ in the equation (24) the collineation L becomes

$$L: \qquad p' = -\frac{1+\sqrt{3}i}{2}\left(p - \sqrt{3}\right).$$

The condition that a collineation,

I The triangles a, a' and a'' form the $(3,3)$ configuration, (a). See Kantor's article Über die Configurationen (3,3) mit den Indices 8, 9, Sitzungsberichte [illegible]

[illegible]

In a [illegible] [illegible] [illegible] [illegible]
[illegible] [illegible] is

$$|a| = 1.$$

[illegible]

[illegible]

and the angle of [illegible] [illegible]
[illegible] the relation

$$\cos\theta = [illegible].$$

In the combination [illegible]

$$\alpha = -\frac{1 \pm \sqrt{3}\,i}{2},$$

$$\alpha' = -\tfrac{1}{2},$$

$$\beta = -\tfrac{\sqrt{3}}{2}.$$

Hence,

$$|\alpha| = \left|\frac{1 \pm \sqrt{3}\,i}{2}\right| = 1,$$

and

$$\tan\theta = \sqrt{3}.$$

[illegible]

Evidently $\theta \neq 60°$, as we must have $3\theta \equiv 2\pi$.

The collineation L is, then, a pure rotation through $-120°$ about the fixed point K.

The collineation L^{-1} sends the circumcircle into the apolar circle, and so, as this collineation is of course also a pure rotation, the circumcircle and apolar circle have equal radii.

It is easy to see that this is only the case when $s^2 = 3$.

For, since L^{-1} is

$$z' = \tfrac{1}{4}(B + iA)z + S, \qquad (25).$$

the map-equation of the apolar circle is

$$z = \tfrac{1}{4}(B + iA)t + $$

where t runs along the unit circle.

The radius of the apolar circle is, then,

$$\left|\frac{B+A}{4}\right| = \sqrt{\frac{B^2+A}{16}} = \sqrt{\frac{-[illegible]B}{16}}.$$

If this is to be unity, we have

$$B = -2,$$

and, hence,

$$s^2 = [illegible].$$

Hence whenever the radii of the circum- and apolar-circles are the same, there exists a collineation having fixed points at infinity and one of the intersections of these circles which sends the triangle into an inscribed triangle, and whose cube is the identical

Collinear line.

In this form the condition may be easily put in projective language. The line joining the points K and L, that is, the line $I_1 = 0$, is the common chord of the circum- and apolar-circles. If these circles have equal radii, this line bisects the join of their centres. It, then, follows that the centres of the circum- and apolar circles are apolar to the lines $I_1 = 0$ and $D = 0$.

We shall now find the condition in terms of the cotangents of the angles of the triangle.

The circumcircle and apolar circle have respectively the equations

$$\sum (c_2 + c_3) b_2 b_3 = 0,$$

and

$$\sum c_1 b_1^2 = 0.$$

In brief, the equations of these circles are

$$\sum (c_2 + c_3)^2 \xi_1^2 - 2(c_1^2 + 1) \xi_2 \xi_3 = 0$$

and

$$\sum c_2 c_3 \xi_1^2 = 0.$$

The centres are the polar points of the line at infinity as to the circles.

That is, the circumcentre is

$$\sum (1 - c_2 c_3) \xi_1 = 0,$$

and the apolar-centre, (i.e. the orthocentre), is

$$\sum c_2 c_3 \xi_1 = 0.$$

The product of these two equations is

$$\sum c_2 c_3 (1 - c_2 c_3) \xi_1^2 + c_1 (c_2 + c_3 - 2 c_1 c_2 c_3) \xi_2 \xi_3 = 0.$$

We wish, then, this degenerate conic to be apolar to the lines $I_1 = 0$ and $D = 0$; i.e. to

$$I_1 D \equiv \sum c_1 b_1^2 + (c_2 + c_3) b_2 b_3 = 0.$$

The condition, then, is

$$\sum 2 c_1 c_2 c_3 (1 - c_2 c_3) + c_1 (c_2 + c_3)(c_2 + c_3 - 2 c_1 c_2 c_3) = 0,$$

which immediately reduces to

$$\sum c_1 (c_2 + c_3)^2 = 0. \qquad (26).$$

The circumcircle in points is of course always apolar to the absolute circle in lines. It is easy to see that the equation

(26) is the condition that the circumcircle in lines is apolar to the apolar circle in points.

For the circumcircle in lines is

$$\sum (c_2+c_3)^2 \xi_1^2 - 2(c_1^2+1)\xi_2\xi_3 = 0,$$

and the apolar circle in points is

$$\sum c_1 x_1^2 = 0.$$

These two conics are apolar if the equation (26) is satisfied. That is, if

$$\sum c_1 (c_2+c_3)^2 = 0. \qquad (26).$$

We have, then, the following theorem:–

If a circumconic and an apolar conic of a given triangle are apolar both ways, there exists a collineation, whose cube is the identical collineation, which leaves fixed

points at three of the intersections of the two conics and which sends the triangle into an inscribed perspective triangle, the centre of perspective being the point into which the collineation sends the fourth intersection of the conics.

Sec. (3). The determination of the loci of a point k, such that it will project two triangles upon any plane, S, into triangles satisfying certain given conditions.

1. Let us have in space of three dimensions three points a, b and c and three lines [illegible]

lie in a plane.

If we project a, b, c, α, β and γ from a point p upon a plane ξ, we shall have in this plane a 3-point, $a'b'c'$, and a 3-line, $\alpha'\beta'\gamma'$. The point p may evidently be so taken that the 3-point and the 3-line in the plane ξ satisfy any projective invariant conditions we choose to impose upon them.

We may, since all the conditions we shall consider are projective, take ξ as the plane of the 3-line.

Let the edges of the fundamental tetrahedron in the plane

$$b_4 = 0$$

be the lines α, β and γ.

[illegible]

points a, b and c from o upon the plane

$$b_4 = 0.$$

The coördinates of any point on the join of a and b are of the form

$$a_i + \lambda b_i .$$

If this point is on the plane

$$b_4 = 0,$$

we have

$$a_4 + \lambda b_4 = 0.$$

The point a' has, then, the coör-dinates

$$A_{41}, A_{42}, A_{43}, A_{44},$$

where

$$A_{4i} = \begin{vmatrix} b_4 & b_i \\ a_4 & a_i \end{vmatrix}.$$

The coördinates of the point α', referred to the triangle formed by the points (1000), (0100) and (0010) are, then, A_{41}, A_{42} and A_{43}. Also, since we are projecting upon the plane of the 3-line, $\alpha\beta\gamma$, the new 3-line, $\alpha'\beta'\gamma'$, will be identical to the original one, and will make up the lines of the triangle of reference considered above.

The locus of o, such that it will project the 3-point $\alpha'\beta'\gamma'$ upon the 3-line, $\alpha\beta\gamma$, into a 3-point and 3-line satisfying any one of the invariant conditions considered in Sec. 1), will evidently be found by substituting the coördinates of the projected points and lines in that invariant.

2. We shall now determine the forms assumed by the invariants D, I, and N, when the a, b and c of Eq.(1) are replaced by the projections of the points a, b and c of space upon the plane $b_4 = 0$, and when the lines $\alpha\beta\gamma$ are taken as the sides of the fundamental triangle in this plane.

We have, then,

$$D = \sum_a A_{41}(B_{42}C_{43} - B_{43}C_{42}).$$

Now,

$$B_{42}C_{43} = b_2 c_3 b_4^2 - b_2 c_4 b_3 b_4 - b_4 c_3 b_2 b_4 + b_4 c_4 b_2 b_3.$$

And $B_{43}C_{42}$ is, evidently, formed from this by interchanging b and c.

Then,

$$B_{42}C_{43} = b_3 c_2 b_4^2 - b_4 c_2 b_3 b_4 - b_3 c_4 b_2 b_4 + b_4 c_4 b_2 b_3.$$

which follows

$$B_{42}\check{C}_{43} + B_{43}C_{42} = (b_2c_3 + b_3c_2)d_4^2 - (b_4c_2 + b_2c_4)d_3d_4$$
$$- (b_3c_4 + b_4c_3)d_2d_4 + 2b_4c_4d_2d_3, \qquad (27).$$

and

$$B_{42}C_{43} - B_{43}C_{42} = (b_2c_3 - b_3c_2)d_4^2 + (b_4c_2 - b_2c_4)d_3b_4$$
$$+ (b_3c_4 - b_4c_3)d_2d_4. \qquad (28).$$

We have, then,

$$D = \sum_a (d_4a_1 - d_1a_4)\left[d_4^2(b_2c_3 - b_3c_2) + d_3d_4(b_4c_2 - b_2c_4) + d_2d_4(b_3c_4 - b_4c_3)\right]$$

$$= \sum_a d_4\left[-a_4(b_2c_3 - b_3c_2)d_1 + a_1(b_3c_4 - b_4c_3)d_2 + a_1(b_4c_2 - b_2c_4)d_3 + a_1(b_2c_3 - b_3c_2)b_4\right]$$
$$- \sum_a a_4b_4\left[(b_4c_2 - b_2c_4)d_3d_1 + (b_3c_4 - b_4c_3)d_1d_2\right].$$

The second summation is identically zero, and the first is the determinant of the points d, a, b and c with the negative sign.

That is,

$$D = -p_4^2 \, |p\,a\,b\,c|. \qquad (29).$$

The factor p_4^2 evidently refers to the plane ξ and not to the plane of the [illegible].

From Sec. (11), we have

$$I_1 = 3 \sum_a a_1 (b_2 c_3 + b_3 c_2).$$

Hence,

$$I_1 = 3 \sum_a A_{41} (B_{42} C_{43} + B_{43} C_{42}).$$

From the equation (27), this becomes

$$I_1 = 3 \sum_a (p_4 a_1 - p_1 a_4) \Big[p_4^2 (b_2 c_3 + b_3 c_2) - p_3 p_4 (b_4 c_2 + b_2 c_4) - p_2 p_4 (b_3 c_4 + b_4 c_3) + 2\, b_2 p_3 b_4 c_4 \Big],$$

$$= 3 \sum_a p_4^3 a_1 (b_2 c_3 + b_3 c_2) - p_4^2 p_1 a_4 (b_2 c_3 + b_3 c_2) - p_4^2 p_2 a_1 (b_3 c_4 + b_4 c_3) - p_4^2 p_3 a_1 (b_4 c_2 + b_2 c_4)$$

$$-2 b_1 b_2 b_3 a_4 b_4 c_4 + 2 b_2 b_3 b_4 a_1 b_4 c_4 +$$

$$b_3 b_4 b_1 a_4 (b_4 c_2 + b_2 c_4) + b_4 b_1 b_2 a_4 (b_3 c_4 + b_4 c_3).$$

And, finally,

$$I_1 \equiv 3 \sum_a \Big[a_1 (b_2 c_3 + b_3 c_2) b_4^3 - b_4^2 \sum_{123} a_4 (b_2 c_3 + b_3 c_2) b_1$$
$$+ 2 b_4 \sum_{123} a_1 b_4 c_4 b_2 b_3 - 2 a_4 b_4 c_4 b_1 b_2 b_3 \Big]. \quad (30).$$

Again referring to Sec. VI, we have

$$N \equiv \sum_a (b_2 c_3 + b_3 c_2) \Big[(c_3 a_1 + c_1 a_3)(a_1 b_2 + a_2 b_1) -$$
$$(c_1 a_2 + c_2 a_1)(a_3 b_1 + a_1 b_3) \Big].$$

The surface corresponding to N is then, of the sixth degree. It is, however, easily seen that it must contain the plane $b_4 = 0$ counted twice. A geometrical proof of this is as follows.

We are to consider [illegible]

$$\Delta D I_1 + \lambda N = 0.$$

As we have already remarked, the factor b_4^2 appearing in Δ pertains to the plane upon which we project and not to the plane of the 3-line. Then, as we are only considering projective properties, this factor must also be contained in N.

Projectively, there is no loss of generality if we take the points b and c as the centroid and fourth vertex of the fundamental tetrahedron. When b and c are so taken a calculation of the invariants gives us.

$$\begin{aligned} N \equiv b_4^2 \sum_{123} (a_2^2 - a_3^2)\, b_4^2 b_1^2 + 2a_1(a_2 - a_3)\, b_4^2 b_2 b_3 \\ + 2(a_2 + a_3)\big[(a_3 - a_4) b_2 + (a_4 - a_2) b_3\big] b_4 b_1^2 \\ + 2a_4(a_2 - a_3)\, b_1^2 b_2 b_3. \qquad (31). \end{aligned}$$

$$I_1 \equiv -3 \sum_{123} (a_2 + a_3) b_4^2 b_1 - 2(a_1 + a_4) b_2 b_3 b_4$$
$$+ 2 a_4 b_1 b_2 b_3 . \qquad (32).$$

$$D \equiv - b_4^2 \sum_{123} (a_2 - a_3) b_1 . \qquad (33).$$

And, hence,

$$\Delta D I_1 \equiv 3 b_4^2 \sum_{123} (a_2^2 - a_3^2) b_4 b_1^2 - 2 a_1 (a_2 - a_3) b_4 b_2 b_3$$
$$- 2(a_2 - a_3)\left[(a_3 + a_4) b_2 + (a_2 + a_4) b_3\right] b_4 b_1^2$$
$$+ 6 a_4 (a_2 - a_3) b_1^2 b_2 b_3 . \qquad (34).$$

From the geometrical meaning of the equations (11) of section (1), it is obvious that the surfaces correspond-ing to the invariants

$$\Delta D I_1 + \lambda N = 0$$

and

$$\Delta' D' I_1' + \lambda N' = 0$$

may be considered as

surface referred to two distinct tetrahedra. That is, whatever we can say for the points a, b, c in the surface corresponding to the latter invariant, we can also say for the roots of α, β, γ in that corresponding to the former.

3. <u>The Cubic Surface Corresponding to I_1.</u>

From the equations (11) of Sec. (1) and (31) and (34) of this Section, we have

$$I_1' \equiv -6\, b_4 \sum_{123}^{3} (a_2^2 - a_3^2)\, b_4 b_1^2 + 4 a_1 (a_2 - a_3)\, b_4 b_2 b_3$$
$$+ 2\left[a_3(a_3 + 2a_2) - a_4(a_2 + 2a_3)\right] b_1^2 b_2$$
$$+ 2\left[a_4(a_3 + 2a_2) - a_2(a_2 + 2a_3)\right] b_1^2 b_3 \qquad (35).$$

---

1. See Cayley's <u>Collected Mathematical Papers</u>, Vol. <u>VI</u>, p. 418 ff.; Salmon-Fiedler, <u>Analytische Geometrie des Raumes</u>, Chap. V, especially the type (8) of p. 374, p. 397 ff. and 410-411.

We also have

$$I_1 \equiv 3 \sum_a \Big[a_1(b_2c_3+b_3c_2)b_4^3 - \sum_{123} a_4(b_2c_3+b_3c_2)b_4^2 b_1 + 2\sum_{123} a_1 b_4 c_4 b_2 b_3 b_4 - 2 a_4 b_4 c_4 b_1 b_2 b_3 \Big] \quad (30).$$

In the discussion of this surface, it will be found convenient to take as the fourth vertex of the fundamental tetrahedron the second polar of the plane of the 3-line as to the 3-point. This point is

$$\sum_a \sum_1^4 a_1 b_4 c_4 \xi_1 = 0.$$

Upon identifying this with the point (0001), we have

$$\sum_a a_i b_4 c_4 = 0, \qquad i = 1, 2, 3.$$

If, now, we let

$$W_i = \sum_a a_4(b_j c_k + b_k c_j), \quad i = 1, 2, 3.$$

$$W_4 = -\sum_a a_1(b_2c_3+b_3c_2),$$

and finally

$$M = 6\, a_4 b_4 c_4,$$

we have

$$-\tfrac{1}{3} I_1 \equiv W_b\, b_4^2 + M\, b_1 b_2 b_3 = 0. \qquad (36).$$

In this equation there are no terms of the second or third degree in b_1, b_2 or b_3. That is, the vertices of the 3-line are nodal points of the surface.

On the other hand, these points lie on the surface I_1', but have perfectly ordinary tangent planes. Hence, the points of the 3-point are ordinary points on the surface I_1'.

We shall now consider the lines

We shall arrange the lines on the surface in classes in the way adopted by Cayley in the article cited above. By an axis we shall mean the join of two nodes, by a transversal, a line meeting an axis in a point other than a node and by a ray a line through a single node. All other lines on a cubic surface are called mere lines, but in the surface we are considering all of the lines are axes, transversals or rays.

From the form of the equation (36), it is evident that the planes

$$p_4 = 0 \quad \text{and} \quad w_0 = 0$$

[illegible]

$$p_1 = 0, \; p_2 = 0 \; [illegible] = 0$$

in lines on the surface.
The lines in the plane

$$p_4 = 0$$

are evidently the three axes.
The lines in the plane

$$m_2 = 0$$

are the Conversely every line lies in a plane containing an axis.
The equation

$$p_4 = \lambda p_2$$

represents any plane through the axis $\overline{12}$. Elimination by between the equation of this plane and that of the cubic surface [illegible]
[illegible]

$$p_3 \left[w_1 p_1 + w_2 p_2 + p_3 (w_3 + \lambda w_4) \right] \lambda^2 + M p_1 p_2 = 0.$$

This, then, is the equation of a cone whose vertex is the point (0001) and whose directrix is the curve of intersection of the plane and the cubic. It may, however, equally well be considered as the equation of the projection of the curve from (0001) upon the opposite plane. From this point of view it represents a conic which evidently degenerates when and only when the plane

$$p_4 = \lambda p_3$$

cuts the cubic in three lines.
In general there are five planes through a line on a cubic surface which cut the surface again in

degenerate conic. If, then, we express the condition that the conic we have found degenerates, we should have a quintic in λ, whose roots give the required pair planes.

The conic degenerates, if

$$\begin{vmatrix} 0 & u & \lambda^2 w_1 \\ u & 0 & \lambda^2 w_2 \\ \lambda^2 w_1 & \lambda^2 w_2 & 2\lambda^2(w_3 + \lambda w_4) \end{vmatrix} = 0.$$

that is, is

$$\lambda^2 \left[w_1 w_2 \lambda^2 - u w_4 \lambda - u w_3 \right] = 0.$$

The term of the fifth degree in λ does not appear, and so infinity must be counted as one of the roots.

[illegible] are

$$\frac{\frac{1}{2}\left(M w_4 \pm \sqrt{M^2 w_4^2 + 4 M w_1 w_2 w_3}\right)}{w_1 w_2}.$$

The denominators of the fractions giving the four other roots are symmetric in the subscripts 1, 2 and 3.

[illegible]

The root 0 gives the planes of the three axes.

The root ∞ gives the plane of an axis and a transversal. This plane, in fact, meets the surface in the axis $\overline{12}$ counted twice and in the transversal

$$w_b = 0, \quad b_3 = 0.$$

The plane, then, touches the surface along the axis 12.

[illegible]

the axis 13 [illegible] cut the surface again in two lines, are

$$P p_3 = w_1 w_2 x_4 \quad \text{and} \quad P' p_3 = w_1 w_2 x_4 .$$

Similarly the planes

$$P p_1 = w_2 w_3 x_4 \quad \text{and} \quad P' p_1 = w_2 w_3 x_4$$

cut the surface in the axis $\overline{23}$ and in two other lines.

By direct substitution, it is easily seen that the P-plane through one axis cuts the P'-plane through another axis in a line on the surface. It follows, then, that the lines in the P- and P'-planes pass in by [illegible] through the three vertices, and, hence, that they are all [illegible]!

Let the plane determined [illegible]

The plane through the nodal rays is called a bi-radial plane.
The two rays through the node (0010) are

$$P x_1 - w_2 w_3 x_4 = P' x_2 - w_3 w_1 x_4 = 0$$

and

$$P' x_1 - w_2 w_3 x_4 = P x_2 - w_3 w_1 x_4 = 0.$$

It is, then, easily seen that the bi-radial plane through this node is

$$w_1 x_1 + w_2 x_2 + w_4 x_4 = 0.$$

That is, the bi-radial plane through our node cuts the cubic again in the transversal which lies in a plane containing the other two nodes.

Thus p-planes

$Pp_1 = w_2 w_3 p_4$, $Pp_2 = w_3 w_1 p_4$ and $Pp_3 = w_1 w_2 p_4$

meet in the point $(w_2 w_3, w_3 w_1, w_1 w_2, P)$. Similarly, the three P'-planes meet in the point $(w_2 w_3, w_3 w_1, w_1 w_2, P')$. The join of these two points evidently passes through the point [illegible]. That is, through the polar point of the plane of the 3-line as to the 3-point.

It is easily seen that this line meets the plane of the 3-line in the polar point of the plane

$$w_p = 0$$

as to the meets of the 3-line, and [illegible]

[illegible]

is the polar point of the plane of the 3-line and B the [illegible] of the three transversals.

It is also obvious that the three bi-radial planes meet in a point on the line we have been considering.

4. In this paragraph we shall call attention to a few of the more obvious facts relating to the quartic surfaces which correspond to the invariants

$$\Sigma D I_1 + \lambda \Lambda = 0.$$

From the equations (31) and (34), we see that the equations of all of these quartic surfaces have the form of [illegible]

degrees. It follows, then, that they all have nodal points at the points of the 3-point and at the meets of the 3-line.

It is also at once evident that the joins of the 3-point and the lines of the 3-line are lines on all of the quartics, and that the intersection of the planes of the 3-line and 3-point is a line on these surfaces.

Among all of the invariants

$$\Delta D I_1 + \lambda N = 0$$

there is a single one which expresses a mutual relation between the six points of the 3-line and 3-point: the vanishing of the [illegible]

that the six points lie on a conic. The corresponding value of λ is -3.

It follows, then, that the surface

$$\Delta D I_1 - 3K = 0$$

is symmetrical with regard to the points of the 3-point and the nests of the 3-line.

From the equations ([illegible]) — (34), the equation of this surface[1] is

$$\sum_{123} a_1(a_2-a_3)b_2 b_3 b_4^2 + a_3(a_2-a_4)b_1^2 b_2 b_4 + a_2(a_4-a_3)b_1^2 b_3 b_4 - a_4(a_2-a_3)b_1^2 b_2 b_3 = 0$$

The six points are nodal points and

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1. Weddle's surface. See Cayley's *Collected Mathematical Papers*, Vol. VII, p. 160 ff.; and a paper by H. S. White, *Annals of Mathematics*, July 1902, p. 151.

the fifteen lines joining these points are lines of the surface. That these lines lie on the surface is immediate from the meaning of the invariant. I.e., any point on the join of two of the points projects these into the same point on any plane ξ. The six points are then projected into but five points on ξ, and through these we can, of course, always draw a conic.

The plane

$$\lambda_4 = 0$$

cut the surface where

$$b_1 b_2 b_3 \sum_{123} a_4 (a_2 - a_3) b_1 = 0.$$

That is, in the lines of the 3-line and in the intersection of the planes

$$b_4 = 0 \quad \text{and} \quad \sum a_4(a_2 - a_3)\, b_1 = 0.$$

The latter plane is, however, the plane of the 3 point. Hence the plane of any three of the six points meets the plane of the remaining three in a line on the quartic. It is also easily seen that the plane of any three of the points touches the surface where, and only where, the join of these three points meets the plane of the other three.

The rational cubic curve through the six points lies on the surface. This is evident since a rational cubic is projected from any of its points upon any plane, into a conic.

5. All of the quartic surfaces corresponding to the invariants

$$\Delta D I_1 + \lambda N' = 0$$

form a pencil of surfaces, which pass through a curve of the sixteenth order.

If the point ε, from which we project, lies on this curve, the projected 3-point and 3-line satisfy all of the invariant conditions considered in Sec. (1). That is, they form a 3-point and a 3-line of the type considered in Sec. (2).

The curve of intersection contains the lines of the 3-line, the joins of the 3-point and the intersection of the planes of the 3-line and 3-point. The remaining part of

the curve is a curve of the ninth degree, which is in fact the intersection of the two cubic surfaces corresponding to the invariants

$$I_1 = 0 \quad \text{and} \quad I_1' = 0.$$

This curve evidently has double points at each point of the [illegible]-point and at the vertices of the 3-line.

It seems very improbable that the curve breaks up into two or more of lower degree.

Let the vertices of the 3-line be denoted by the letters A, B and C, and let the surface I_1 cut the line $\overline{BC}$ in a third point a_1, and similarly let I_1' cut the line $\overline{BC}$ in a third point A_1. The curve, then, passes through

the [illegible] points [illegible]

way.

It is easy to see that the rational cubic through a, b, c, A, B, and C is not a part of the curve we are considering. This may be shown analytically, but it is also geometrically evident. For were this cubic a part of the intersection of I_1 and I_1', the six projected points would be in a mutual relation, which is not the case.

The argument just used also shows that the rational cubic through the points a_1, b_1, c_1, A_1, B_1, and C_1 does not form a part of the curve. For, if it were, the remaining part would be a sextic curve having double points at the points of the 3-point and at the [illegible]

the 3-line. This, again, would imply a resultant relation between the six projected points.
The only remaining cubics which treat the 3-point and the nests of the 3-line alike, are those through a, b, c, A_1, B_1, C_1 and a_1, b_1, c_1, A, B, C. If the curve contained one of these it would of necessity contain the other also. The remaining part would, then, be the rational cubic through a, b, c, A, B and C. But we have seen that this is not a part of the curve.
Further, the curve cannot break up into two quartics, as then L_1 and L_1' would have a common line.

Vita

I, John [illegible], was born at Albany, New York, in November, 1877. I prepared for college at the Albany Academy, and took the degree of A.B. at Williams College in 1899. The following October I entered the Johns Hopkins University as a graduate student in Mathematics, Physics, and Philosophy. I have held a scholarship and fellowship, and was a student assistant in the year 1901–1902.

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